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ABSTRACT

This research addressed itself to the development of a novel teaching-learning scheme for non-trivial, first-degree equations in one unknown, based on an analysis of the notions and misconceptions of junior high students. The approach was tried in a "teaching experiment" involving six case studies of students of various abilities in grades 7 and 8. Results indicate that this method of introducing equations is effective both at the junior high level and also with older remedial students. Other findings lead to a new perspective on students' concept of equality and left-to-right sequencing of operations. (Author/MK)

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CONSTRUCTING MEANING FOR NON-TRIVIAL EQUATIONS

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CONSTRUCTING MEANING FOR NON-TRIVIAL EQUATIONS

This research addressed itself to 1) the development of a novel teaching-learning scheme for non-trivial first degree equations in one unknown (i.e., equations of the form $ax \pm b = cx \pm d$ where the unknown occurs on both sides, making it unlikely that the equation can be solved by inspection) based on an analysis of the notions and misconceptions of junior high school students and on a consideration of relevant theories of learning; 2) the field-testing and refinement of this teaching-learning scheme by means of a "teaching experiment" which consisted of individual interviews with six students from grades 7 and 8.

Statement of the Problem

Despite efforts by educators to teach students both "what equations are" and "how to solve them", recent research (e.g., Davis and Cooney, 1979; Matz, 1979) has shown that students continue to have difficulties in this area. Two of the major stumbling blocks seem to involve the use of letters and the interpretation of the equal sign.

1) The use of letters

It is commonly assumed that, because students have worked with problems such as $2 \times \square = 18$ in elementary school, they can handle algebraic equations with letters. However, boxes don't get operated on or moved about as letters do; boxes are merely placeholders (Wagner, 1977). They are filled in, like blanks - a connotation which letters lack.

Some students initially fail to realize that a letter represents a number and thus encounter difficulty operating with letters which have no specific referent. The following excerpt from one of Davis' (1975) interviews provides some evidence of this problem:

I: Now we multiply the right hand side by x. What do we get?
Henry: How can we multiply by x when we don't know what x is?

According to Davis, "It seemed clear that Henry (a very bright 7th grader) was not recognizing that x was, in fact, some number.

That this is not an isolated problem is indicated by the results of other such studies. For example, Firth (1975) tested 17 pupils, aged 14 and 15 years, on their ability to handle the symbolism of algebra. One of the questions he asked was: "If x is any number, write the number which is 3 more than x." Of the 8 pupils (41%) who had answered incorrectly, it is

interesting to note that 5 of them had each selected a particular value for x and had given an answer 3 bigger than the value they had chosen. It seems that many students are unable to cope with expressions where unevaluated operations are held in suspension (Collis, 1978). They would not consider "x + 3" to be a suitable answer for Firth's question above.

Others consider letters as objects which they can disregard. Evidence of this is given by Küchemann (1978) who tested 3000 secondary school children in Britain (aged 13 - 15 years) on their understanding of variable. One of his questions, "Add 4 onto 3n", produced the following percentages of incorrect responses: 78% for 13-year-olds, 64% for 14-year-olds, 59% for 15-year-olds. The dominant wrong answer (31%) was $7n$ instead of $3n + 4$. According to Küchemann, "the elements in the item are simply joined in the most immediately obvious way, without reference to what the elements might represent: 3 and 4 give 7 and the n is just tagged on the end. Thus, essentially the letter is ignored, and in fact a further 16% simply gave the answer 7, so ignoring n entirely".

Other students do not realize that the choice of letter used in an equation is quite arbitrary. Wagner (1977) presented 72 individual students (aged 12, 14, 17 years) with two equations,

$$7 \times w + 22 = 109 \qquad 7 \times n + 22 = 109$$

and asked which solution would be larger, w or n. Only 50% of the 12-year-olds answered correctly that "the solution would be the same". Some of the incorrect answers included, "can't tell until both equations are solved; "the solution of the first equation is greater than the second one because w comes later than n in the alphabet".

2) Interpretation of the equal sign

Davis (1975) has pointed out that "the ability to use the equal sign in several different ways is one of the cognitive demands" of working with equations in high school. However, most students beginning junior high school have a fairly primitive notion of the use of the equal sign. It is one which they carry with them during most of their years in elementary school - as indicated by the results of the following studies.

From his work with children at the early elementary level, Ginsburg (1977) has shown that children's understanding of the symbol for equality refers to actions. They do not view $3 + 5 = 8$ as an arithmetic equivalence but rather operationally, as their reading, "3 and 5 make 8", indicates.

Ginsburg has also stated that many first and second graders, when asked how to read $\square = 3 + 4$, would answer "Blank equals 3 plus 4", but then add: "It's backwards! Do you read backwards?" They would then change it to $4 + 3 = \square$ and say: "You can't go, 7 equals 3 + 4". He also found that many children cannot read sentences that express relationships like $3 = 3$ or $4 = 4$.

It can be argued that these notions reflect the kind of instruction that these children have received. One might then assume that later exposure to equalities involving the commutative and associative properties might broaden the elementary school child's notion of the equal sign. However, this does not seem to be the case. Behr, Erlwanger, and Nichols (1976) did a study with children in grades 1 to 6 which investigated their view of equality sentences. When Behr et al. asked sixth graders about the meaning of "3 = 3", a typical response was: "This could mean '0 minus 3 equals 3'" and wrote $0 - 3 = 3$. Upon further questioning, this same subject changed $3 = 3$ to $6 - 3 = 3$ and then to $7 - 4 = 3$. Behr and his colleagues have emphasized that even "sixth graders seem to view the equal sign as a '<u>do something signal</u>'".

For most students entering junior high school, the equal sign is still associated with the <u>answer</u>. It is telling that seventh graders, when asked for an example showing the use of the equal sign, usually limit themselves to examples involving an operation on the left side and the result on the right (Herscovics and Kieran, in press).

The difficulties which students encounter when using literal symbols and also the limited notions of the equal sign which they bring with them from their arithmetic must both be taken into account in any teaching-learning scheme on algebraic equations.

<u>Summary of Teaching-Learning Scheme</u>

Such considerations have led to the development of the following scheme which builds on the student's arithmetic and transforms it step by step in an <u>explicit</u> construction. The scheme which was based on the constructivist learning theory of Piaget (1973), Bruner's (1966) theory on modes of representation, and the didactic model of Herscovics (1979), aimed at 1) constructing meaning for the concept of equation and 2) laying the groundwork for the eventual justification of the algebraic operations used in the solution of equations.

1) Constructing meaning for the concept of equation

The objectives of the first part of the teaching-learning scheme were to extend the notion of the equal sign and to introduce the concept of an algebraic equation.

i) Extending the notion of the equal sign

In order to have the student construct explicitly meaning for the new mathematical form (an algebraic equation), it was considered essential to use his already-acquired arithmetic knowledge as a starting point. Thus the learner would be building upon well established mental structures and would transform these very gradually. This would avoid any possible gap which might prevent him from assimilating the new material (Flavell, 1963).

The first transformation that would be necessary would involve expanding the student's notion of the equal sign beyond the "process on the left side - result on the right side" interpretation. This would be done by having the student construct arithmetic equalities, initially with one operation on each side, e.g.,

$2 \times 6 = 4 \times 3$ (the same operation)

and then $2 \times 6 = 10 + 2$ (different operations).

The student would then go on to constructing equalities with two operations on each side, and then to multiple operations on each side,

e.g., $7 \times 2 + 3 - 2 = 5 \times 2 - 1 + 6$.

These would be given the name "arithmetic identities" in order to reserve the term "equation" for use in the algebraic sense.

The reason for extending the notion of the equal sign to include multiple operations on both sides was to provide a foundation for the later construction of meaning for non-trivial algebraic equations (which have multiple operations on both sides). If this expansion were not done first, the student would be bringing with him into the study of algebraic equations the idea that the result is always on the right side of the equal sign. Thus, equations such as $3x + 5 = 26$ might fit in with his existing notions, but $3x + 5 = 2x + 12$ would not. Not only would the presence of this multiple operation on the right side be foreign to him, but also seeing it for the first time within the context of an algebraic equation would add to the cognitive strain. Therefore, extending the notion of the equal sign within the framework of arithmetic equalities prior to the introduction of algebraic equations was considered essential in the construction of meaning for non-trivial equations.

ii) Introducing the concept of equation

The next step would involve taking one of the student's arithmetic identities,

e.g., $7 \times 2 - 3 = 5 \times 2 + 1$

and hiding any one of the numbers. The hiding would be done at first by a finger:

$7 \times$ $- 3 = 5 \times 2 + 1$,

then by a box:

$7 \times \square - 3 = 5 \times 2 + 1$,

and finally by a letter:

$7 \times a - 3 = 5 \times 2 + 1$.

Thus an equation would be defined as

AN ARITHMETIC IDENTITY WITH A HIDDEN NUMBER.

By following these three modes of representation (which parallel Bruner's enactive, iconic, and symbolic stages), the student could acquire an intuitive understanding of the meaning of equation and then gradually transform this to an understanding of the form of an algebraic equation. Thus, his algebra would be anchored in his arithmetic. The letter hiding the number would be called an unknown - a term which corresponds closely to the idea of a hidden number.

At this stage the student would construct several equations from his or her own arithmetic identities. In view of Wagner's findings (described earlier), it would seem necessary that he or she be encouraged to choose many different letters. It would be equally important for him or her to realize that a given arithmetic identity could lead to the construction of many different equations:

e.g., $3 \times 7 + 3 = 25 - 1$ yielding $a \times 7 + 3 = 25 - 1$

$3 \times b + 3 = 25 - 1$

$3 \times 7 + 3 = c - 1$ etc.

This kind of variety in building equations would prevent students from imposing unnecessary restrictions on the concept of equation.

The question of hiding more than one number would come up (or be brought up) soon afterward. Since students would not be solving equations at this point, but would still be constructing them, there would be no reason to restrict them to equations with just one letter. Therefore, when they would begin to hide more than one number, they would be told of the convention

that the same letter could be used in an equation more than once, as long as it was used to hide the same number; otherwise one would have to use two different letters.

Thus, in hiding the same number twice - one occurrence on the left side of the equal sign and the other occurrence on the right - the student would be constructing non-trivial algebraic equations,

e.g., $2 \times 3 + 7 = 5 \times 3 - 2$ leading to $2 \times c + 7 = 5 \times c - 2$.

By following this construction, there would be no doubt in the student's mind that the letter represents a number. Furthermore, this approach would allow the student to avoid the pitfalls associated with the use of letters and the interpretation of the equal sign which were described earlier.

2) Laying the groundwork for the eventual justification of the algebraic operations used in the solution of equations

Arithmetic identities would also be used to lay the groundwork for the eventual justification of the algebraic operations used in the solution of equations. To justify these operations, many teachers introduce the idea of a scale. However, the scale does not lend itself readily to addition and subtraction of arbitrary rational numbers nor to the more complex operations of multiplication and division. Arithmetic identities, on the other hand, are not subject to these physical restrictions. In addition, any operation performed on an arithmetic identity would be immediately verifiable by the student. Furthermore, since students would, by this time, be defining equations as arithmetic identities with a hidden number, the operations performed on arithmetic identities could be transferred to operations on equations.

This part of the teaching-learning scheme would involve taking one of the student's arithmetic identities and "adding-on" some number to one side,

e.g.,
$$5 \times 3 - 4 = 11$$
$$5 \times 3 - 4 + 7 = 11 \ldots$$

The student would then be asked how he could make it into an arithmetic identity again. It was expected that the student would see the need for performing an operation on the other side to restore the balance. When restricted to the use of the same operation, he would be led to doing the same thing on both sides, i.e.,

$$5 \times 3 - 4 = 11$$
$$5 \times 3 - 4 + 7 = 11 + 7$$

A similar approach would be used for examples involving the insertion of the other operations of subtraction, multiplication, and then division, leading finally to the general rule:

WHATEVER YOU DO TO ONE SIDE, DO TO THE OTHER SIDE.

This rule could then be connected to the algebraic processes used in solving equations by handling simultaneously an arithmetic identity and an equation derived from it (Herscovics and Kieran, in press)

e.g., $7 \times \boxed{3} + 4 = 9 \times \boxed{3} - 2$ $\quad 7x + 4 = 9x - 2$

The operations to be performed at each step in solving the equation would be verified explicitly with the "arithmetic identity" on the left. It might be pointed out, however, that the ability to justify the rules should not be mistakenly interpreted as a strategy for solving equations. As with all problems whose solution involves multiple steps, a pupil may very well be able to justify each individual step without achieving the overview needed to develop solution strategies. (These, however, are beyond the scope of this study.)

Methodology

The methodology chosen for the field-testing of this teaching-learning scheme was a version of the Soviet "teaching experiment" - a methodology which, according to Menchinskaya (1969), "is directed at disclosing the very process of learning as it takes place under the influence of pedagogy." In contrast to Piagetians who assign very little significance to the role of instruction in the development of the child, Soviet psychologists study the development of thinking under the changing conditions of instruction.

1) The "teaching experiment"

The Soviet "teaching experiment" can take many forms. One of them - the experiment of instruction - involves the introduction of preliminary stages of study in which the initial information and abilities needed to master the new material are ascertained and arranged in a hierarchy. During the instruction which follows the planning stages, the experimenter investigates the learner's thinking and seeks to discover the changes which are occurring. Menchinskaya describes this form as being very advantageous in that it allows the experimenter to follow the particular changes in the mental processes of the same pupils on an individual basis.

Under another form - experiments of assessment, Soviet psychologists investigate children who have already mastered certain concepts, on their ability to use them to solve problems. In both of these forms, the investigator seeks to observe the dynamics of the learning process. Under yet another mode - the testing mode - the investigator tries to use different methods to discover which of these promotes the most effective mastery of information.

When the Soviet researchers wish to look into the child's process of acquiring and using concepts, they use the individual interview. When they wish to confirm the results of individual experiments on a wider basis, they use the group experiment. However, according to Kantowski (1979), the statistical analysis of quantitative data is usually of less concern than the analysis of qualitative data.

2) Version used

The version of the Soviet "teaching experiment" used for this study was the "experiment of instruction". The dynamic nature of the methodology permitted this researcher to inquire, during individual interviews, into the subject's mental activity and to study the immediate effects of the planned instruction. This methodology also provided the flexibility to alter, whenever deemed necessary, the planned teaching sequence and to deviate from the set of prepared questions to pursue any unforeseen or interesting responses which might come out during the interviews. In fact, subjects were encouraged to "think aloud" and to interrupt the teaching at any time with questions or comments. Furthermore, these questions and comments allowed for the continuous refinement of the teaching-learning scheme.

This methodology required the interviewer to be both teacher and observer - a task that was, at times, difficult to manage. Nevertheless, the methodology proved to be most rewarding in terms of the richness of data gathered. It had the advantage of yielding immediate feedback on the effectiveness of any teaching sequence and provided for the repeated testing of hypotheses. If the analysis following each interview indicated that one of the subject's responses warranted further investigation, the interviewer was able to probe for more detail at the next session with the same subject. The opportunity of meeting each subject over several sessions produced enough data to result in case studies. Another reason for the wealth of data drew directly from the nature of the processing involved in the "constructive learning"

situation. The subjects were conscious of many of the cognitive changes which were taking place and could verbalize them, in constrast to some problem-solving studies where subjects are often unaware of their cognitive processing.

3) Data source

Six subjects (aged 12-14 years) were involved in this study, three from grade 7 and three from grade 8. They were from different schools and represented a fairly broad range of ability - most of them having been loosely classified by their math teachers (another characteristic of the Soviet "teaching experiments") as being strong, average, or weak in mathematics. None of them had as yet been exposed to any systematic treatment of algebraic equations in school.

Subjects had one interview session per week, for five to eight weeks, each one lasting from 25-45 minutes. The first two sessions dealt with those aspects of the teaching-learning scheme which are described in this paper; the remaining sessions were devoted to the solution of equations and are beyond the scope of this presentation. The interview sessions were audio-taped and later transcribed in their entirety for purposes of analysis. Both the subjects' written work and their verbal protocols were used as data.

Analysis of the Protocols

Since the teaching-learning scheme had been drawn up in advance according to a specific plan, the subjects' protocols could be divided, for purposes of analysis, into segments which corresponded to the main subdivisions of the teaching scheme.

Since most of the same questions had been asked of all subjects, it was possible to look at the responses to each question and to groups of questions across subjects. These were examined for common thinking patterns and for signs of individual differences. An example follows below which indicates the answers which subjects gave to two successive questions in the initial phase of the first interview.

Interviewer: What does the equal sign mean to you?
M: The same thing.
G: It means what the answer is.
C: It is, like after the equal sign is when ... it's hard to say ... after the equal sign is the answer.
Pa: A total of numbers is going to come after that.
Pi: You have to have two or more numbers before them and then they work in some way with a sign, multiplication, or addition and then whatever the sum or product is, the equal sign comes before it. Meaning that whatever the two numbers mean with the sign, it equals so and so.

Interviewer: Could you give me an example where you use the equal sign?
B: $6 \times 3 = 18$, $9 + 17 = 26$
M: $4 + 3 = 7$
G: At the end of an equation ... before your answer ... $5 + 3 = 8$ ... $5 \times 3 = 15$
C: In multiplication, subtracting ... $2 \times 3 = 6$
Pa: $2 + 5 = 7$ ... $4 + 3 = 7$
Pi: $6 + 2 = 8$, $8 - 2 = 6$, $8 \cdot 2 = 16$, $16 \div 2 = 8$

The responses given above offer evidence that these junior high school students held the notion that "after the equal sign is the answer." In fact, the following excerpt shows the difficulty that one student had in considering an equality which did not have the answer on the right side:

(These questions were put to subject M immediately following his response of $4 + 3 = 7$ above.)

Interviewer: And what does $4 + 3 = 6 + 1$ mean to you?
M: We don't do that kind of thing in school. We just put $4 + 3 = 7$.
Interviewer: Do you always put the answer after the "equals"?
M: Yeh.
Interviewer: Is that the only time you use the equal sign?
M: Yeh.

The protocols were also analyzed subject by subject. This provided a general overview of each subject - including information on stability of responses, strengths and weaknesses, and individual reactions to particular parts of the presentation. Since the methodology used afforded the flexibility to go beyond the immediate answer - whether it was verbal or written - there were questions in some protocols which were not found in others. For example:

Interviewer: Now if I look at your example [$5 \times 3 = 15$], I see one operation here, on this side - the multiplication operation - and I see the answer on the right side.
G: Yes
Interviewer: Can you use the equal sign with one operation on each side?
G: What do you mean?
Interviewer: Which operation have you used in that equality?
G: Multiplication.
Interviewer: And how many operations do you have written here?
G: One ... two 5×3 .
Interviewer: You have two numbers, but one operation. Do you have an operation on this side?
G: No
Interviewer: Is it possible for you to write an equality with an operation on each side?
G: You mean like $5 \times 3 = 15$ and $15 = 10 + 5$?

Interviewer: Would you write that down?
G: $5 \times 3 = 15 = 10 + 5$
Interviewer: You have two equal signs.
G: Yes.
Interviewer: Is it possible to write here something similar to that, but with the equal sign written just once? An operation on one side, an operation on the other, with just one equal sign.
G: You mean two operations with one equal sign?
Interviewer: Uh, hm.
G: $5 \times 3 = 10 + 5$
Interviewer: You have an equality with one operation on the left side, and one operation on the right side. Is that true?
G: Yes.
Interviewer: Why?
G: 'Cuz the answer is the same.

This excerpt points out that questions which may seem obvious or even trivial to us may not be understood ("What do you mean?") by the learner. There is often the need to clarify a question, to go beyond the seeming lack of communication. The rigid adherence to a prescribed set of questions may only yield a wrong answer, whereas informal questioning, supplementing the standardized interview, may reveal underlying competence and at the same time allow the learner to advance his thinking.

Results and Conclusions

Some of the thinking patterns which were demonstrated by the subjects during their individual interviews are summarized below. A few pedagogical implications of these observations are also included.

1) On the equal sign

Initially, subjects' notion of the equal sign had been restricted to a single operation on the left side and the result on the right. This confirmed the findings of Behr et al. (1976) that "there is no evidence to suggest that children change in their thinking about equality as they get older and progress to upper grades; in fact, the evidence seems to be to the contrary" (p.10).

As described in the teaching-learning scheme, their notion of the equal sign was expanded to include multiple operations on both sides. However, at no time was there any evidence to suggest that they ever viewed a string of arithmetic operations in the static sense as being "another name for a number." For example, $5 \times 7 + 3$ was not another name for 38, rather it had a value of 38. They interpreted arithmetic identities such as $3 + 5 - 2 = 5 + 1$ in terms of both sides being equal because they have the same value. Further-

more, equations such as $2x + 3 = 4x + 1$ were described by subjects: "If we knew what number x is, then 2 times that number plus 3 has the same value as 4 times that number plus 1."

These comparisons that subjects were eventually able to make between left and right sides of the equal sign suggest that the equality symbol was being seen at this later stage more as a relational symbol than as a "do something signal." The right side, by this time, did not have to contain the answer, but rather could be some expression that had the same value as the left side.

2) On sequential processing

It was also observed that subjects were processing strings of arithmetic operations sequentially from left to right. When constructing multi-operation arithmetic equalities, they would write them down, operation by operation, as they were thinking of them, and would keep a running total as they went along (Kieran, 1979). Thus, according to their "running-total heuristic", $2 + 3 \times 5$ would have a value of 25 rather than 17 - the result if evaluated according to standard conventions where multiplication takes precedence over addition. Their natural inclination to evaluate arithmetic strings from left to right seemed to override the conventions which they had all learned at some time prior.

The evidence for this sequential way of viewing mathematical operations strengthens Laborde's (1978) findings that junior high school students tend to define "dynamically" rather than "statically", that is, they give an explanation of an event in terms of the operations which are used. Furthermore, the subevents leading up to this event are described in the order in which they occur, and usually by means of specific examples.

3) On the notion of solution

Even before discussing the notion of solution, several subjects indicated an awareness of this concept. The process of hiding a number seemed to carry inherently the reverse process of uncovering. Thus, the concept of solution, as the "number hidden by the letter", was grasped even before solution processes were introduced.

The notion of multiple solutions also came about with this construction when students asked about hiding more than one number. One student hid both 2's in the arithmetic identity $3 + (1 \times 4) = 10 \div 2 + 2$ to obtain $3 + (1 \times 4) = 10 \div c + c$, and immediately noticed that 5 was another possible

solution. He seemed quite surprised to discover that the unknown could be replaced by a number other than the one he had hidden. Showing that two distinct arithmetic identities (i.e., $3 + (1 \times 4) = 10 + 2 + 2$ and $3 + (1 \times 4) = 10 + 5 + 5$) can lead to the same equation seemed to be a satisfactory explanation pending the introduction of the theory of quadratic equations.

Another subject decided to hide both 3's in $2 + 3 = 3 + 2$ to yield $2 + a = a + 2$, only to discover that he now had an equation in which he could replace the letter by "any number." Subjects' reactions to this type of equation suggest that the search for other such examples can provide a good opportunity for discussion of the axioms of our number systems. It seems t at for the many students who may never have understood the expression of these axioms in symbolic form, these postulates can now become more meaningful when presented in the context of constructing equations (Herscovics & Kieran, in press).

Educational Importance of the Study

This study is important for two main reasons: the value of the "teaching experiment" methodology as a tool for researching cognitive processing; the contribution of this novel yet simple approach to the teaching of equations at the junior high school level.

1) Importance of the methodology

The methodology that was used in this study, a version of the Soviet "teaching experiment", proved to be most effective in disclosing many of the cognitive processes involved in acquiring the concept of an algebraic equation. As is obvious from the few excerpts included within this paper, there is more to developing a teaching-learning scheme than amalgamating an analysis of: learning theories, error studies, structure of the subject material. Testing a teaching-learning outline in individual interviews provides essential information on unforeseen cognitive difficulties which could possibly develop into errors later on or even prevent the learner from achieving the necessary constructions.

The value of the individual interview is, however, not restricted to research involving a teaching component. Herbert Ginsburg (1977), among others, has used the technique of the individual interview for his outstanding case studies in which he investigated how children understand and do arithmetic. Repeated interviews with the same children can provide very

rich data not only on the processes of cognition, but also on the stability of acquired knowledge.

2) Importance of the teaching-learning scheme

The teaching-learning scheme served as an excellent introduction to the topic of algebraic equations and allowed students to avoid several of the more common stumbling blocks associated with the symbolic representations of number and equality. It is an approach which was found to be accessible not only to all the subjects in the study, but also to classes of mature adults taking high school algebra at university. It is not suggested, however, that this method for introducing algebraic equations replace all existing presentations (e.g., the "think of a number" and word problem approaches), but rather be used just prior to them. For no one approach is sufficient or even advisable in the learning of any complex cognitive task. This approach is a strong beginning.

The introduction to equations, which is suggested here, reflects the distinction between mathematical content (concepts, rules, relationships) and mathematical form (the notation and symbolism used to express the content). By initially defining an equation as an arithmetic identity with a hidden number, learners can assimilate the new mathematical idea before it becomes frozen in a mathematical form new to them. In teaching mathematics the distinction between content and form has important pedagogical implications, for, as pointed out by Byers (1978), "teaching for understanding requires that the continuity of mathematical content be demonstrated to the student during, and prior to, the introduction of the new mathematical form."

References

Behr, M., Erlwanger, S., & Nichols, E. How children view equality sentences. PMDC Technical Report No. 3. Florida State University, 1976. (ERIC Document Reproduction Service No. ED 144 802)

Bruner, J.S. Toward a theory of instruction. Cambridge: Harvard University Press, 1966.

Byers, V. Essays in mathematics education, part 2. Unpublished manuscript, Concordia University, Montreal, 1978.

Collis, K. Operational thinking in elementary mathematics. In J. Keats, K. Collis, & G. Halford (Eds.), Cognitive development. New York: John Wiley, 1978.

Davis, R.B. Cognitive processes involved in solving simple algebraic equations. The Journal of Children's Mathematical Behavior, Vol. 1, No. 3, (Summer, 1975).

Davis, E.J. & Cooney, T.J. Identifying errors in solving certain linear equations. Paper presented at the annual meeting of the American Educational Research Association, San Francisco, April 1979.

Flavell, J. The developmental psychology of Jean Piaget. New York: Van Nostrand, 1963.

Firth, D.E. A study of rule dependence in algebra. Unpublished M. Phil. thesis, University of Nottingham, 1975.

Ginsburg, H. Children's arithmetic. New York: Van Nostrand, 1977.

Herscovics, N. A learning model for some algebraic concepts. In K. Fuson & W. Geeslin (Eds.), Explorations in the modeling of the learning of mathematics. Columbus: ERIC/SMEAC, 1979.

Herscovics, N. & Kieran, C. Constructing meaning for the concept of equation. The Mathematics Teacher, in press.

Kantowski, M.G. The teaching experiment and Soviet studies of problem solving. In L. Hatfield & D. Bradbard (Eds.), Mathematical problem solving: Papers from a research workshop. Columbus: ERIC/SMEAC, 1979.

Kieran, C. Children's operational thinking within the context of bracketing and the order of operations. Proceedings of Third International Conference for the Psychology of Mathematics Education. Warwick, England, 1979.

Küchemann, D. Children's understanding of numerical variables. Mathematics in School, (U.K.), Vol. 7, No. 4, September 1978.

Laborde, C. Relations arithmétiques - aspect statique - aspect dynamique. Educational Studies in Mathematics, 9, 1978.

Matz, M. Towards a process model for high school algebra errors. Paper presented at the Conference on Cognitive Processes in Algebra, University of Pittsburgh, July 1979.

Menchinskaya, N. The psychology of mastering concepts: Fundamental problems and methods of research. In J. Kilpatrick & I. Wirszup (Eds.), Soviet studies in the psychology of learning and teaching mathematics (Vol. I). Stanford: SMSG, 1969.

Piaget, J. To understand is to invent. New York: Viking Press, 1973.

Wagner, S. Conservation of equation, conservation of function, and their relationship to formal operational thinking. Unpublished doctoral dissertation, New York University, 1977.